

Higher Derivative Fermionic Field Equation in the First Order Formalism

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Abstract

The generalized Dirac equation of the third order, describing particles with spin 1/2 and three mass states, is analyzed. We obtain the first order generalized Dirac equation in the 24-dimensional matrix form. The mass and spin projection operators are found which extract solutions of the wave equation corresponding to pure spin states of particles. The density of the electromagnetic current is obtained, and minimal and non-minimal (anomalous) electromagnetic interactions of fermions are considered by introducing three phenomenological parameters. The Hamiltonian form of the first order equation has been obtained.

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1 Introduction

The standard model (SM) of electroweak interactions does not explain the mass spectra of leptons and quarks, and contains many free parameters. One of the most fundamental problems is the mass generation of fermions. Therefore, it is important to investigate the unified description of quarks and leptons and understand the origin of fermion masses. The mass generation mechanism is beyond the SM and is needed for the deeper understanding of lepton-quark families.

We consider here the toy model based on the generalized Dirac equation of the third order (see [1]¹) possessing flavor S_3 symmetry, for particles with three nonzero mass states. One can speculate that this equation may be applied for unified description of lepton-quark generations (see [2], [3], [4], [5])

¹In the work [1], authors investigated the case of two nonzero masses and one zero mass for the unified description of e , μ -leptons and neutrino.

for the case of two mass states). According to the Barut approach [2], such equations may be treated as effective equations for partly “dressed” fermions, and such schemes represent the non-perturbative approach to quantum electrodynamics.

It should be noted that higher derivative (HD) field theories have been investigated already. Some examples include generalized electrodynamics [6], renormalizable gravity theory in four dimensions [7] and others. Therefore, it is motivated to investigate HD field theories in four dimensions.

In this paper, we represent the third order (in derivatives) equation in the form of the first order generalized Dirac equation, obtain solutions in the form of projection operators and consider minimal and non-minimal electromagnetic interactions.

The paper is organized as follows. In Sec. 2, we define the generalized Dirac equation of the third order for fermions, obtain projection operators extracting solutions of the HD equation, and analyze a propagator of fields. The first order generalized 24-component Dirac equation is derived in Sec. 3. It is proven that the mass spectrum of the 24-component matrix equation obtained and the third order 4-component equation is the same. We obtain mass and spin projection operators in Sec. 4. These projection operators extract pure spin states of particles. Sec. 5 is devoted to introducing minimal and non-minimal electromagnetic interactions of fermions. In this section, we find also the electromagnetic current density. We postulate the matrix equation with three phenomenological parameters characterizing anomalous electromagnetic fermion interactions and derive the quantum-mechanical Hamiltonian. A conclusion is made in Sec. 6. In appendixes, we obtain useful relations of 24×24 -matrices of the wave equation and spin matrices.

The system of units $\hbar = c = 1$ is chosen and euclidian metric is used (see [8]).

2 Field Equation of Third Order for Fermions

We postulate the third order (in derivatives) field equation describing fermions:

$$(\gamma_\mu \partial_\mu + m_1) (\gamma_\nu \partial_\nu + m_2) (\gamma_\alpha \partial_\alpha + m_3) \psi(x) = 0, \quad (1)$$

where $\partial_\nu = \partial/\partial x_\nu = (\partial/\partial x_m, \partial/\partial(it))$, $\psi(x)$ is a bispinor; m_1, m_2, m_3 are masses of fermions. As usual, repeated indices imply a summation. The Dirac Hermitian matrices γ_μ obey the commutation relations $\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu =$

$2\delta_{\mu\nu}$. It is obvious, that there are three independent solutions of Eq. (1) corresponding to masses m_i ($i = 1, 2, 3$). The bispinor $\bar{\psi}(x) = \psi^+(x)\gamma_4$ (ψ^+ is the Hermitian-conjugated bispinor) satisfies the equation

$$\bar{\psi}(x) \left(\gamma_\mu \overleftarrow{\partial}_\mu - m_1 \right) \left(\gamma_\nu \overleftarrow{\partial}_\nu - m_2 \right) \left(\gamma_\alpha \overleftarrow{\partial}_\alpha - m_3 \right) = 0, \quad (2)$$

where the derivatives $\overleftarrow{\partial}_\mu$ act on the left-standing function. The discrete symmetries of the wave function $\psi(x)$ directly follow from the Dirac theory.

Eq. (1) for positive energy, in momentum space, is given by

$$(i\hat{p} + m_1)(i\hat{p} + m_2)(i\hat{p} + m_3)\psi(p) = 0, \quad (3)$$

where $\hat{p} = \gamma_\mu p_\mu$, $p_\mu = (\mathbf{p}, ip_0)$. For antiparticles one should make the replacement $p \rightarrow -p$. From Eq. (3), one obtains the dispersion equation as follows:

$$(p^2 + m_1^2)(p^2 + m_2^2)(p^2 + m_3^2) = 0, \quad (4)$$

The projection operator extracting solutions to Eq. (3), corresponding to positive energy, is given by

$$\Lambda_+ = \frac{(-i\hat{p} + m_1)(-i\hat{p} + m_2)(-i\hat{p} + m_3)}{2[m_1 m_2 m_3 - p^2(m_1 + m_2 + m_3)]}. \quad (5)$$

It is easy to verify that the projection operator (5) obeys the necessary equation $\Lambda_+^2 = \Lambda_+$. It should be noted that the projection operator Λ_+ is the solution to Eq. (3) for any momentum p obeying the dispersion relation (4), i.e. the momentum p is not specified here. The projection operator Λ_+ corresponds to positive energy, and at the replacement $p \rightarrow -p$, to negative energy of particles.

In the momentum space the propagator of the HD field is

$$\begin{aligned} \Delta(p) &= \frac{1}{(i\hat{p} + m_1)(i\hat{p} + m_2)(i\hat{p} + m_3)} \\ &= \frac{(-i\hat{p} + m_1)(-i\hat{p} + m_2)(-i\hat{p} + m_3)}{(p^2 + m_1^2)(p^2 + m_2^2)(p^2 + m_3^2)}. \end{aligned} \quad (6)$$

The propagator (6) can be represented as a sum of Dirac propagators

$$\Delta(p) = \frac{A(-i\hat{p} + m_1)}{(p^2 + m_1^2)} - \frac{B(-i\hat{p} + m_2)}{(p^2 + m_2^2)} + \frac{C(-i\hat{p} + m_3)}{(p^2 + m_3^2)}, \quad (7)$$

where the coefficients A , B , and C are given by

$$\begin{aligned} A &= \frac{1}{(m_2 - m_1)(m_3 - m_1)}, & B &= \frac{1}{(m_2 - m_1)(m_3 - m_2)}, \\ C &= \frac{1}{(m_3 - m_2)(m_3 - m_1)}. \end{aligned} \quad (8)$$

Eq. (7) is valid only in the case $m_1 \neq m_2 \neq m_3$, i.e. masses of the field are not degenerated. In the case $m_3 > m_2 > m_1$, the coefficients A , B , and C are positive, and the second propagator in Eq. (7), corresponding to the mass m_2 , has the “wrong” sign $(-)$. This means that the HD field possesses two physical states with masses m_1 , m_3 , and one Weyl ghost with the mass m_2 . The ghost gives the negative contribution to the energy [3], [9], and, as a result, the Hamiltonian is not bounded from below. Therefore, to formulate the quantum field theory one needs to introduce the indefinite metrics [3]. As the propagator (7) is a sum of Dirac propagators, the renormalization of the theory can be performed with the help of the standard procedure of QED.

3 24-Component First Order Relativistic Wave Equation

For a convenience, we rewrite Eq. (1) in the form

$$\left(\gamma_\mu \partial_\mu \partial_\nu^2 + a \gamma_\mu \partial_\mu + b \partial_\nu^2 + c \right) \psi(x) = 0, \quad (9)$$

where $\partial_\nu^2 = \partial_m^2 - \partial_0^2$ ($m = 1, 2, 3$, $\partial_0 = \partial/\partial t$). The coefficients

$$a = m_1 m_2 + m_1 m_3 + m_2 m_3, \quad b = m_1 + m_2 + m_3, \quad c = m_1 m_2 m_3 \quad (10)$$

are invariant under the permutation symmetry S_3 .

Now we reformulate the third order Eq. (9) in the form of the first order relativistic wave equation. It is useful for Lagrangian formulation of the theory, obtaining the conserved currents, and different calculations. Let us introduce the 24-dimensional function

$$\Psi(x) = \{\Psi_A(x)\} = \begin{pmatrix} \psi(x) \\ \tilde{\psi}(x) \\ \psi_\mu(x) \end{pmatrix}, \quad (11)$$

where $A = 0, \tilde{0}, \mu$, and

$$\begin{aligned}\Psi_0(x) &= \psi(x), & \Psi_{\tilde{0}}(x) &= \tilde{\psi}(x) \equiv \frac{1}{a} \partial_\mu^2 \psi(x), \\ \Psi_\mu(x) &= \psi_\mu \equiv -\frac{b}{a} \partial_\mu \psi(x).\end{aligned}\tag{12}$$

The wave function $\Psi(x)$ represents, in Eq. (12), the direct sum of bispinor $\psi(x)$, bispinor $\tilde{\psi}(x)$, vector-bispinor $\psi_\mu(x)$, and transforms under the Lorentz group as

$$[(1/2, 0) \oplus (0, 1/2)] \oplus [(1/2, 0) \oplus (0, 1/2)] \oplus \{(1/2, 1/2) \otimes [(1/2, 0) \oplus (0, 1/2)]\}.$$

To obtain the first order relativistic wave equation from Eq. (9), we introduce the elements of the entire algebra $\varepsilon^{A,B}$ (see, for example, [10]) obeying the multiplication rule and having matrix elements as follows:

$$\varepsilon^{M,A} \varepsilon^{B,N} = \delta_{AB} \varepsilon^{M,N}, \quad (\varepsilon^{M,N})_{AB} = \delta_{MA} \delta_{NB}, \tag{13}$$

$A, B, M, N = 0, \tilde{0}, \mu$ ($\mu = 1, 2, 3, 4$). The matrix $\varepsilon^{M,N}$ consists of zeros and only one element is unity, where row M and column N cross.

Using Eq. (11)-(13), Eq. (9) takes the form of the first order equation

$$\begin{aligned}\partial_\mu \left(\varepsilon^{\mu,0} + \varepsilon^{\tilde{0},\mu} - \varepsilon^{0,\mu} + \varepsilon^{0,0} \gamma_\mu + \varepsilon^{0,\tilde{0}} \tilde{\gamma}_\mu \right)_{AB} \Psi_B(x) \\ + \left(\frac{a}{b} \varepsilon^{\mu,\mu} + \frac{c}{a} \varepsilon^{0,0} + b \varepsilon^{\tilde{0},\tilde{0}} \right)_{AB} \Psi_B(x) = 0.\end{aligned}\tag{14}$$

It is implied that γ -matrices act on the bispinor indexes. There is a summation over repeated indices in subscripts and superscripts of the matrices. Now we introduce the matrices

$$\Gamma_\mu = \left(\varepsilon^{\mu,0} + \varepsilon^{\tilde{0},\mu} - \varepsilon^{0,\mu} \right) \otimes I_4 + \left(\varepsilon^{0,0} + \varepsilon^{0,\tilde{0}} \right) \otimes \gamma_\mu, \tag{15}$$

$$M = \frac{c}{a} P_0 + b \tilde{P}_0 + \frac{a}{b} P_1, \tag{16}$$

$$P_0 = \varepsilon^{0,0} \otimes I_4, \quad \tilde{P}_0 = \varepsilon^{\tilde{0},\tilde{0}} \otimes I_4, \quad P_1 = \varepsilon^{\mu,\mu} \otimes I_4. \tag{17}$$

Unit four-dimensional matrix I_4 acts on bispinor indexes and matrices $\varepsilon^{M,N}$ ($M, N = 0, \tilde{0}, \mu$) act on scalar and vector indexes. It should be mentioned

that matrices P_0, \tilde{P}_0, P_1 are projection matrices obeying the relation $P^2 = P$. The M is the diagonal and Hermitian matrix, $M = M^+$, but the matrix Γ_μ is not Hermitian matrix.

Taking into account Eq. (11)-(13), Eq. (14) becomes

$$(\Gamma_\mu \partial_\mu + M) \Psi(x) = 0. \quad (18)$$

Eq. (18) obtained, is the relativistic 24-component wave equation of the first order and is convenient for different applications.

Now, we derive the mass spectrum of Eq. (18). In the frame of references where a particle is at the rest, Eq. (18), in the momentum space, becomes

$$(p_0 \Gamma_4 - M) \Psi(p_0) = 0, \quad (19)$$

where p_0 is the energy of a particle. Here we use one particle (quantum mechanical) interpretation of Eq. (18) and do not discuss the second quantized theory. Masses of particles (see [12]) are given by the relation $m_i = \lambda_i^{-1}$, where λ_i are eigenvalues of the matrix $M^{-1} \Gamma_4$ or $m_i = \bar{\lambda}_i$, where $\bar{\lambda}_i$ are eigenvalues of the matrix $M \Gamma_4^{-1}$. From Eq. (15), (16), one may find expressions

$$M^{-1} = \frac{a}{c} P_0 + \frac{1}{b} \tilde{P}_0 + \frac{b}{a} P_1, \quad (20)$$

$$K \equiv M^{-1} \Gamma_4 = \left(\frac{b}{a} \varepsilon^{4,0} + \frac{1}{b} \tilde{\varepsilon}^{0,4} - \frac{a}{c} \varepsilon^{0,4} \right) \otimes I_4 + \left(\frac{a}{c} \varepsilon^{0,0} + \frac{a}{c} \tilde{\varepsilon}^{0,0} \right) \otimes \gamma_4, \quad (21)$$

With the help of Eq. (13), it is easy to verify that the matrix K obeys the equation as follows:

$$cK^3 I_6 \otimes \gamma_4 - aK^2 + bK I_6 \otimes \gamma_4 - \Lambda = 0, \quad (22)$$

where

$$I_6 \otimes \gamma_4 = \left(\varepsilon^{0,0} + \tilde{\varepsilon}^{0,0} + \varepsilon^{\mu,\mu} \right) \otimes \gamma_4, \quad (23)$$

$$\Lambda = P_0 + \tilde{P}_0 + \varepsilon^{4,4} \otimes I_4. \quad (24)$$

The matrix Λ is the projection matrix, so that $\Lambda^2 = \Lambda$. It follows from Eq. (22), that as the matrix $I_6 \otimes \gamma_4$ possesses eigenvalues ± 1 , the matrix Λ possesses eigenvalues 1, eigenvalues of the matrix K , $\lambda = \{\lambda_i\}$, obey the equation

$$\pm c\lambda^3 - a\lambda^2 \pm b\lambda - 1 = 0. \quad (25)$$

As a result, taking into consideration the relation $m_i = 1/\lambda_i$, the mass spectrum of Eq. (18) is given by solutions of the equation

$$m_i^3 \mp bm_i^2 + am_i \mp c = 0. \quad (26)$$

Positive solutions to Eq. (26), m_i , are connected with the coefficients a , b , and c , possessing the flavor S_3 symmetry, by relationships (10). So, the mass spectrum of Eq. (18) and Eq. (9) is the same.

As the diagonal mass matrix M is not proportional to unit matrix, it is convenient for some applications to get the equivalent form of Eq. (18). Multiplying Eq. (18) by the non-singular matrix

$$N = P_0 + \frac{c}{ab}P_0 + \frac{cb}{a^2}P_1,$$

and introducing the new matrix

$$\bar{\Gamma}_\mu = N\Gamma_\mu = \left(\frac{cb}{a^2}\varepsilon^{\mu,0} + \frac{c}{ab}\varepsilon^{\tilde{0},\mu} - \varepsilon^{0,\mu} \right) \otimes I_4 + \left(\varepsilon^{0,0} + \varepsilon^{0,\tilde{0}} \right) \otimes \gamma_\mu, \quad (27)$$

Eq. (18) transforms to the standard form of the relativistic wave equation

$$(\bar{\Gamma}_\mu \partial_\mu + m) \Psi(x) = 0. \quad (28)$$

We took into consideration the equalities

$$NM = \frac{c}{a}I_{24}, \quad m \equiv \frac{c}{a} = \frac{m_1 m_2 m_3}{m_1 m_2 + m_1 m_3 + m_2 m_3},$$

where I_{24} is the unit 24×24 -matrix, and m is a reduced mass. Notice that the wave function $\Psi(x)$ remains the same (Eq. (11)) in Eq. (18),(28).

4 Mass and Spin Projection Operators

In the momentum space Eq. (28) becomes

$$(i\check{p} + m) \Psi(p) = 0, \quad (29)$$

where $\check{p} = p_\mu \bar{\Gamma}^\mu$. One may verify, with the help of Eq. (A6)-(A7) (see Appendix A), that the \check{p} obeys the matrix equation as follows:

$$\check{p}(\check{p}^2 + m^2) \left[c^2 \check{p}^4 - m^2 \check{p}^2 (p^2 d + c^2) - m^4 p^6 \right] = 0, \quad (30)$$

where $d = m_1^2 m_2^2 + m_1^2 m_3^2 + m_3^2 m_2^2$. From Eq. (30), we find the projection operator

$$\Pi_+ = \frac{\check{p}(\check{p} + im) [c^2 \check{p}^4 - m^2 \check{p}^2 (p^2 d + c^2) - m^4 p^6]}{2m^6 (p^6 - p^2 d - 2c^2)}. \quad (31)$$

It is not difficult to verify, with the help of Eq. (30) that the projection operator (31) obeys the equation $\Pi_+^2 = \Pi_+$. To find the projection operator corresponding to negative energy, Π_- , one has to make the replacement $p \rightarrow -p$ in Eq. (31). Every column of the matrix Π_+ is a solution to Eq. (29), and after acting on 24-dimensional vector χ , one finds the solution to Eq. (29), $\Psi(p) = \Pi_+ \chi$. The matrix (31) also represents the density matrix for the impure spin state.

To obtain the spin projection operators, we consider the generators of the Lorentz group in the 24-dimensional representation which are given by

$$J_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & J_{\mu\nu}^{(1)} \end{pmatrix} \otimes I_4 + I_6 \otimes J_{\mu\nu}^{(1/2)}, \quad (32)$$

where [10]

$$J_{\mu\nu}^{(1)} = \varepsilon^{\mu,\nu} - \varepsilon^{\nu,\mu}, \quad (33)$$

$$J_{\mu\nu}^{(1/2)} = \frac{1}{4} (\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu). \quad (34)$$

Unit 6-dimensional matrix I_6 is defined in the space $(0, \tilde{0}, \mu)$, and the generators of the Lorentz group in four-dimensional vector and bispinor spaces are $J_{\mu\nu}^{(1)}$, $J_{\mu\nu}^{(1/2)}$, respectively. It is easy to verify that generators (32)-(34) satisfy the commutation relations

$$[J_{\mu\nu}, J_{\alpha\beta}] = \delta_{\nu\alpha} J_{\mu\beta} + \delta_{\mu\beta} J_{\nu\alpha} - \delta_{\nu\beta} J_{\mu\alpha} - \delta_{\mu\alpha} J_{\nu\beta}. \quad (35)$$

In our euclidian metric, the antisymmetric parameters of the Lorentz group ω_{mn} ($m, n = 1, 2, 3$) are real, and ω_{m4} are imaginary. The relativistic form-invariance of Eq. (28) follows from the commutation relation [12]

$$[\bar{\Gamma}_\lambda, J_{\mu\nu}] = \delta_{\lambda\mu} \bar{\Gamma}_\nu - \delta_{\lambda\nu} \bar{\Gamma}_\mu, \quad (36)$$

which is valid for matrices (27), (32).

The operator of the spin projection on the direction of the momentum \mathbf{p} is given by (see, for instance, [10])

$$\sigma_p = -\frac{i}{2|\mathbf{p}|}\epsilon_{abc}\mathbf{p}_a J_{bc}, \quad (37)$$

where $|\mathbf{p}| = \sqrt{p_1^2 + p_2^2 + p_3^2}$, ϵ_{abc} ($a, b, c = 1, 2, 3$) is an antisymmetric tensor Levy-Civita. With the help of Eq. (36), one may verify that the operator of Eq. (28) in the momentum space ($i\bar{\Gamma}_\mu p_\mu + m$) commutes with the spin operator (37): $[i\bar{\Gamma}_\mu p_\mu + m, \sigma_p] = 0$. This guaranties the existence of the common solutions of the Eq. (29) in the momentum space, $\Psi_{s_p}(p)$, and the equation

$$\sigma_p \Psi_{s_p}(p) = s_p \Psi_{s_p}(p), \quad (38)$$

where $s_p = \pm 1/2$. With the aid of Eq. (13), (B5) (see Appendix B), one can verify that the matrix equation

$$\left(\sigma_p^2 - \frac{1}{4}\right)\left(\sigma_p^2 - \frac{9}{4}\right) = 0 \quad (39)$$

is valid. This equation allows us to construct two projection operators corresponding to spin 1/2 and 3/2. Indeed, the representation of the Lorentz group corresponding to the 24-dimensional wave function (11), contains also the spin-3/2 component. It is important to show that the wave equation (28) (or (18)) describes only pure spin-1/2 states. For this purpose, we consider both projection operators corresponding to spin 1/2 and 3/2. Exploring the method [11], one obtains the projection operators

$$P_{\pm 1/2} = \mp \frac{1}{2} \left(\sigma_p \pm \frac{1}{2}\right) \left(\sigma_p^2 - \frac{9}{4}\right), \quad P_{\pm 3/2} = \mp \frac{1}{6} \left(\sigma_p \pm \frac{3}{2}\right) \left(\sigma_p^2 - \frac{1}{4}\right) \quad (40)$$

extracting spin projections $\pm 1/2$ and $\pm 3/2$. One can verify that $P_{s_p}^2 = P_{s_p}$ ($s_p = \pm 1/2, \pm 3/2$). With the help of Eq. (B6), we find the relations

$$P_{\pm 3/2} \check{P} = \check{P} P_{\pm 3/2} = 0, \quad P_{\pm 3/2} \Pi_\pm = \Pi_\pm P_{\pm 3/2} = 0, \quad (41)$$

but $P_{\pm 1/2} \Pi_\pm \neq 0$. Therefore, the wave equation (28) (or (18)) describes only pure spin-1/2 states. Eq. (39), (40) lead to the equation

$$\sigma_p P_{s_p} = s_p P_{s_p}. \quad (42)$$

Eq. (42) shows that the wave function $\Psi_{s_p}(p) = P_{s_p}\chi$, (χ being an arbitrary nonzero 24-dimensional column) is the solution of Eq. (38). The projection operator corresponding to common solutions to Eq. (29), (38) is given by

$$\rho_{\pm 1/2}(p) = \Pi_+ P_{\pm 1/2}. \quad (43)$$

The projection operator (43) extracts states with positive energy of particles and spin projections $\pm 1/2$. The $\rho_{\pm 1/2}$ is also the density matrix for pure spin states ($s_p = \pm 1/2$).

5 Non-Minimal Electromagnetic Interactions of Fermions

Now, we obtain the density of the electromagnetic current. For this purpose, Eq. (2) is rewritten as follows:

$$\partial_\mu \partial_\nu^2 \bar{\psi}(x) \gamma_\mu + a \partial_\mu \bar{\psi}(x) \gamma_\mu - b \partial_\nu^2 \bar{\psi}(x) - c \bar{\psi}(x) = 0. \quad (44)$$

Following the well-known procedure, multiplying Eq. (9) (from left) by $\bar{\psi}(x)$, and Eq. (44) (from right) by $\psi(x)$, and adding them, one finds

$$\begin{aligned} & \bar{\psi}(x) \gamma_\mu \partial_\mu \partial_\nu^2 \psi(x) + \left(\partial_\mu \partial_\nu^2 \bar{\psi}(x) \right) \gamma_\mu \psi(x) \\ & + a \left[\bar{\psi}(x) \gamma_\mu \partial_\mu \psi(x) + \left(\partial_\mu \bar{\psi}(x) \right) \gamma_\mu \psi(x) \right] \\ & + b \left[\bar{\psi}(x) \partial_\nu^2 \psi(x) - \left(\partial_\nu^2 \bar{\psi}(x) \right) \psi(x) \right] = 0. \end{aligned} \quad (45)$$

Identifying the left side of Eq. (45) with $(-ia) \partial_\mu j_\mu(x)$, we obtain the electric current density

$$\begin{aligned} j_\mu(x) &= i \bar{\psi}(x) \gamma_\mu \psi(x) + i \frac{b}{a} \left[\bar{\psi}(x) \partial_\mu \psi(x) - \left(\partial_\mu \bar{\psi}(x) \right) \psi(x) \right] \\ &+ \frac{i}{a} \left[\bar{\psi}(x) \gamma_\mu \partial_\nu^2 \psi(x) + \left(\partial_\nu^2 \bar{\psi}(x) \right) \gamma_\mu \psi(x) - \left(\partial_\nu \bar{\psi}(x) \right) \gamma_\nu \partial_\mu \psi(x) \right. \\ &\quad \left. - \left(\partial_\mu \bar{\psi}(x) \right) \gamma_\nu \partial_\nu \psi(x) + \left(\partial_\nu \bar{\psi}(x) \right) \gamma_\mu \partial_\nu \psi(x) \right], \end{aligned} \quad (46)$$

so that the conservation of the four-vector current density $\partial_\mu j_\mu(x) = 0$ holds. The current $j_\mu(x)$ consists of the usual Dirac current $j_\mu^D(x) = i \bar{\psi}(x) \gamma_\mu \psi(x)$ and additional convective terms.

Substituting the derivatives in Eq. (18), $\partial_\mu \rightarrow D_\mu = \partial_\mu - ieA_\mu$ (A_μ is the four-vector potential of the electromagnetic field), one can obtain the minimal interaction with an electromagnetic field. Now we introduce the non-minimal electromagnetic interaction by considering the matrix equation

$$\left[\Gamma_\mu D_\mu + \frac{i}{2} \mathcal{F}_{\mu\nu} \Gamma_{\mu\nu} (\kappa_0 P_0 + \tilde{\kappa}_0 \tilde{P}_0 + \kappa_1 P_1) + M \right] \Psi(x) = 0, \quad (47)$$

where $\mathcal{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the strength of the electromagnetic field, and $\kappa_0, \tilde{\kappa}_0, \kappa_1$ are parameters which characterize anomalous electromagnetic interactions of fermions, and

$$\begin{aligned} \Gamma_{\mu\nu} &= \Gamma_\mu \Gamma_\nu - \Gamma_\nu \Gamma_\mu \\ &= \left(\varepsilon^{0,\tilde{0}} + \varepsilon^{0,0} \right) \otimes (\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu) + \left[\varepsilon^{\mu,0} + \varepsilon^{\mu,\tilde{0}} \right] \otimes \gamma_\nu \\ &\quad - \left[\varepsilon^{\nu,0} + \varepsilon^{\nu,\tilde{0}} \right] \otimes \gamma_\mu + (\varepsilon^{\nu,\mu} - \varepsilon^{\mu,\nu}) \otimes I_4. \end{aligned} \quad (48)$$

The projection operators P_0, \tilde{P}_0, P_1 obey the relations: $P_0 \tilde{P}_0 = P_0 P_1 = \tilde{P}_0 P_1 = 0$, $P_0 + \tilde{P}_0 + P_1 = I_{24}$. Eq. (47) is the relativistic wave equation for interacting fermions which is form-invariant under the Lorentz transformations. The tensor form of Eq. (47) follows from Eq. (11), (15)-(17), (48):

$$\begin{aligned} \gamma_\mu D_\mu [\psi(x) + \tilde{\psi}(x)] + i \mathcal{F}_{\mu\nu} \gamma_\mu \gamma_\nu [\kappa_0 \psi(x) + \tilde{\kappa}_0 \tilde{\psi}(x)] \\ - D_\mu \psi_\mu(x) + \frac{c}{a} \psi(x) = 0, \end{aligned} \quad (49)$$

$$D_\mu \psi(x) + i \mathcal{F}_{\mu\nu} \gamma_\nu [\kappa_0 \psi(x) + \tilde{\kappa}_0 \tilde{\psi}(x)] + \left(\frac{a}{b} \delta_{\mu\nu} - i \kappa_1 \mathcal{F}_{\mu\nu} \right) \psi_\nu(x) = 0, \quad (50)$$

$$D_\mu \psi_\mu(x) + b \tilde{\psi}(x) = 0. \quad (51)$$

Eq. (49)-(51) can be used for the phenomenological applications in the case of anomalous electromagnetic interactions of fermions. The system of Eq. (49)-(51) defines bispinors $\psi(x)$, $\tilde{\psi}(x)$, and vector-bispinor $\psi_\nu(x)$. Eq. (50) may be considered as a matrix equation for vector-bispinor $\psi_\nu(x)$. Finding $\tilde{\psi}(x)$, and vector-bispinor $\psi_\nu(x)$ from Eq. (51), (50) and substituting them into Eq. (49), one may obtain an equation for bispinor $\psi(x)$. This equation contains the interaction with the anomalous magnetic moments of particles. To figure out the physical interpretation of introduced constants $\kappa_0, \kappa_1, \tilde{\kappa}_0$,

one needs to consider the non-relativistic limit of the theory. We leave this for further investigations.

Let us consider the Hamiltonian form of Eq. (47). This form is convenient for determination of dynamical variables of the wave function $\Psi(x)$ and for second quantization of fields under consideration. Eq. (47) can be rewritten as

$$i\Gamma_4\partial_t\Psi(x) = \left[\Gamma_m D_m + eA_0\Gamma_4 + \frac{i}{2}\mathcal{F}_{\mu\nu}\Gamma_{\mu\nu} \left(\kappa_0 P_0 + \tilde{\kappa}_0 \tilde{P}_0 + \kappa_1 P_1 \right) + M \right] \Psi(x). \quad (52)$$

With the aid of Eq. (13), one may verify that the matrix Γ_4 obeys the equation as follows:

$$\Gamma_4^4 = \Lambda, \quad (53)$$

where the projector operator Λ is given by Eq. (24). To separate the canonical and non-canonical parts of the wave function, we define the projection operator

$$\Pi = 1 - \Lambda, \quad (54)$$

where $1 \equiv I_{24}$ is the unit 24×24 -matrix. Projection operators Λ , Π obey equations

$$\Pi + \Lambda = 1, \quad \Pi\Lambda = \Lambda\Pi = 0, \quad \Lambda^2 = \Lambda, \quad \Pi^2 = \Pi. \quad (55)$$

Introducing the auxiliary wave functions

$$\phi(x) = \Lambda\Psi(x), \quad \chi(x) = \Pi\Psi(x), \quad (56)$$

so that $\phi(x) + \chi(x) = \Psi(x)$, multiplying Eq. (52) by Γ_4^3 , and taking into account Eq. (53), one obtains

$$i\partial_t\phi(x) = eA_0\phi(x) + \Gamma_4^3 \left(\Gamma_m D_m + \sigma + M \right) [\phi(x) + \chi(x)], \quad (57)$$

where we have introduced the operator

$$\sigma = \frac{i}{2}\mathcal{F}_{\mu\nu}\Gamma_{\mu\nu} \left(\kappa_0 P_0 + \tilde{\kappa}_0 \tilde{P}_0 + \kappa_1 P_1 \right). \quad (58)$$

Acting on Eq. (52) with the operator Π , and using the equations

$$\Pi\Gamma_4 = 0, \quad \Pi M = \frac{a}{b}\Pi, \quad \Pi\Gamma_m\Pi = 0,$$

we obtain the equation for auxiliary wave function $\chi(x)$

$$\frac{a}{b}\chi(x) + \Pi\Gamma_m D_m\phi(x) + \Pi\sigma([\phi(x) + \chi(x)]) = 0. \quad (59)$$

Expressing the auxiliary wave function $\chi(x)$ from Eq. (59), and replacing it into Eq. (57), one obtains the equation in the Hamiltonian form:

$$i\partial_t\phi(x) = \mathcal{H}\phi(x), \quad (60)$$

$$\mathcal{H} = eA_0 + \Gamma_4^3\left(\Gamma_m D_m + \sigma + M\right) \left[1 - \left(\frac{a}{b} + \Pi\sigma\right)^{-1} \Pi(\Gamma_m D_m + \sigma)\right].$$

The wave function $\phi(x)$ entering Eq. (60), possesses 12 components, and the auxiliary wave function $\chi(x)$ also has 12 components. Indeed, for the relativistic description of spin-1/2 fields (four components for each field) with three mass states, 12 components are needed, and, therefore, the dynamical wave function $\phi(x)$, possesses 12 components.

At the particular case, for point-like particles, without anomalous interactions, $\sigma = 0$, one finds the Hamiltonian from Eq. (60) as follows:

$$\mathcal{H} = eA_0 + \Gamma_4^3\left(\Gamma_m D_m + M\right) \left(1 - \frac{b}{a}\Pi\Gamma_m D_m\right). \quad (61)$$

6 Conclusion

We have considered the generalized Dirac equation of the third order, describing fermions possessing the permutation symmetry S_3 . The 24-dimensional matrix form of the first order was obtained which is convenient for different applications. Although the dimension of the matrices is high (24×24), we have found minimal equations for matrices. This allowed us to obtain the projection operators extracting states with definite energy and spin projections. The density of the electromagnetic current was obtained, and we have introduced three phenomenological parameters characterizing anomalous non-minimal electromagnetic interactions of fermions. It is of interest to clear up the physical interpretation of constants introduced. This and other questions we leave for further investigations.

Dynamical and non-dynamical variables were separated and quantum-mechanical Hamiltonian was derived. First order matrix wave equations obtained are convenient for second quantization and for calculations of different electrodynamic processes.

Non-trivial differences between this three mass case and the previously studied two mass generalized Dirac equation [5] is in the dimension of matrices of wave equation and corresponding minimal equations. We mention that in this three mass case it is not trivial to obtain the Hermitianizing matrix η in our 24- representation case and corresponding Lagrangian.

The HD field theory considered includes the ghost state which requires to introduce indefinite metrics. It should be noted that this negative attribute appears in many HD field theories.

One of important unanswered questions is the mechanism of mass generations of fermions (see [13] and references there). Here, we have introduced three independent parameters which specify the masses and do not explain their appearance. It seems interesting to follow the Barut [2] idea about the electromagnetic nature of fermion masses. This, however, requires the further investigations of the approach considered.

Appendix A

For a convenience, we represent the matrix \check{p} of Eq. (29) as follows

$$\check{p} = I_{(0)} \otimes \hat{p} + I_{(1)} \otimes I_4, \quad (A1)$$

where

$$I_{(0)} = \varepsilon^{0,0} + \varepsilon^{0,\tilde{0}}, \quad I_{(1)} = p_\mu \left(\alpha \varepsilon^{\mu,0} + \beta \varepsilon^{\tilde{0},\mu} - \varepsilon^{0,\mu} \right), \quad (A2)$$

$$\alpha = \frac{cb}{a^2}, \quad \beta = \frac{c}{ab}, \quad \hat{p} = p_\mu \gamma_\mu. \quad (A3)$$

Using Eq. (13) one may prove simple relationships:

$$\begin{aligned} I_{(0)} I_{(1)} I_{(0)} &= 0, \quad I_{(1)}^3 = -\alpha p^2 I_{(1)}, \quad I_{(1)} I_{(0)} I_{(1)} = \alpha (\beta - 1) I_{(2)}, \\ I_{(2)} &= p_\mu p_\nu \varepsilon^{\mu,\nu}, \quad I_{(2)}^2 = p^2 I_{(2)}, \\ I_{(2)} I_{(0)} &= I_{(0)} I_{(2)} = 0, \quad I_{(1)} I_{(2)} I_{(1)} = p^2 \left(I_{(1)}^2 + \alpha I_{(2)} \right), \\ I_{(2)} I_{(1)}^2 &= -\alpha p^2 I_{(2)}, \quad I_{(0)} I_{(1)}^2 I_{(0)} = \alpha (\beta - 1) p^2 I_{(0)}, \\ I_{(1)}^2 I_{(0)} I_{(1)} &+ I_{(1)} I_{(0)} I_{(1)}^2 = \alpha (\beta - 1) p^2 I_{(1)}. \end{aligned} \quad (A4)$$

One may check, with the help of Eq. (13),(A4), that the minimal equation

$$\check{p}^7 + (2\alpha - 1)p^2\check{p}^5 + \alpha(\alpha - 2\beta)p^4\check{p}^3 - \alpha^2\beta^2p^6\check{p} = 0. \quad (A5)$$

is valid. It should be noted that the dispersion equation (4) can be rewritten as

$$m^6 - (2\alpha - 1)m^4p^2 + \alpha(\alpha - 2\beta)m^2p^4 + \alpha^2\beta^2p^6 = 0. \quad (A6)$$

With the help of Eq. (A6), Eq. (A5) may be represented as follows:

$$\check{p} \left(\check{p}^2 - \frac{m^2}{m_1^2} p^2 \right) \left(\check{p}^2 - \frac{m^2}{m_2^2} p^2 \right) \left(\check{p}^2 - \frac{m^2}{m_3^2} p^2 \right) = 0. \quad (A7)$$

Although the \check{p} is 24×24 -matrix it obeys the simple minimal equation.

Appendix B

The spin operator (37) can be written as

$$\sigma_p = \sigma_p^{(1)} \otimes I_4 + I_6 \otimes \sigma_p^{(1/2)}, \quad (B1)$$

where

$$\sigma_p^{(1)} = -\frac{i}{2|\mathbf{p}|} \epsilon_{abc} \mathbf{p}_a J_{bc}^{(1)}, \quad \sigma_p^{(1/2)} = -\frac{i}{2|\mathbf{p}|} \epsilon_{abc} \mathbf{p}_a J_{bc}^{(1/2)}. \quad (B2)$$

The operators $\sigma_p^{(1)}$, $\sigma_p^{(1/2)}$ correspond to spin-1 and spin-1/2 representations of the Lorentz group. Using Eq. (33), (34), we obtain

$$\left(\sigma_p^{(1)} \right)^2 = \varepsilon^{m,m} - \frac{p_a p_b}{\mathbf{p}^2} \varepsilon^{a,b}, \quad \left(\sigma_p^{(1/2)} \right)^2 = \frac{1}{4}. \quad (B3)$$

so, that the spin operators obey the simple matrix equations

$$\sigma_p^{(1)} \left(\sigma_p^{(1)} - 1 \right) \left(\sigma_p^{(1)} + 1 \right) = 0, \quad \left(\sigma_p^{(1/2)} - \frac{1}{2} \right) \left(\sigma_p^{(1/2)} + \frac{1}{2} \right) = 0. \quad (B4)$$

From Eq. (B1), (B3), one finds

$$(\sigma_p)^2 = \frac{1}{4} + \left(\sigma_p^{(1)} \right)^2 \otimes I_4 + 2\sigma_p^{(1)} \otimes \sigma_p^{(1/2)}. \quad (B5)$$

Using Eq. (B3), (B5), we obtain the useful relation

$$\left[(\sigma_p)^2 - \frac{1}{4} \right] \check{p} = 0. \quad (B6)$$

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